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Probability Density Function for Waves Propagating in a Straight Rough Wall Tunnel

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INTRODUCTION

The radio channel places fundamental limitations on the performance of wireless communication systems in tunnels and caves. The transmission path between the transmitter and receiver can vary from a simple direct line of sight to one that is severely obstructed by rough walls and corners. Unlike wired channels that are stationary and predictable, radio channels can be extremely random and difficult to analyze. In fact, modeling the radio channel has historically been one of the more challenging parts of any radio system design; this is often done using statistical methods.

The mechanisms behind electromagnetic wave propagation are diverse, but can generally be attributed to reflection, diffraction, and scattering. Because of the multiple reflections from rough walls, the electromagnetic waves travel along different paths of varying lengths. The interactions between these waves cause multipath fading at any location, and the strengths of the waves decrease as the distance between the transmitter and receiver increases.

As a consequence of the central limit theorem, the received signals are approximately Gaussian random process. This means that the field propagating in a cave or tunnel is typically a complex-valued Gaussian random process.

ANALYSIS

For this analysis we are considering only high carrier frequencies. It is reasonable to suppose that at any point the received field is far from the source as well.

Let us start by representing a transmitted signal as

$$s(t) = \Re[s_l(t)e^{j2\pi f_c t}], \tag{1}$$

where $s_l(t)$ is the modulated baseband signal and f_c is the carrier frequency, in the GHz range. Since there exist multiple propagation paths, the received signal consists of many signals, each of which is described by a propagation delay and an attenuation factor. Both the propagation delays and the attenuation factors are spatially dependent, as a result of changes in the structure of the medium or boundaries. Thus the received bandpass signal is expressed in the form

$$r(t)=\sum_{n=1}^{N}\alpha_n s(t-\tau_n) \qquad (2)$$

where α_n is the attenuation factor for the signal received on the *nth* path and τ_n is the propagation delay for the *nth* path. We have

$$r(t)=\Re\left\{\left[\sum_{n=1}^{N}\alpha_n e^{-j2\pi f_c\tau_n}s_l(t-\tau_n)\right]e^{j2\pi f_c t}\right\}. \qquad (3)$$

It is apparent from (3) that the received signal consists of a sum of a number of time-variant phasors having amplitudes α_n and phases $2\pi f_c\tau_n$. If $\tau_n=1/f_c$, the phase will change by 2π. Assume $f_c=1\ GHz$, $\tau_n=1\ ns$ will result a 2π phase change, which is a small amount for the delay. If $\tau_n>1ns$ the phase will change $2\pi-\theta_n=\theta_n$.

Let us assume that the cave diameter is 2 meters, for example. It is not difficult to show that there is *2 ns* propagation delay at a location 1 meter away from the transmitter, which is a 4π phase change. If the receiver is located more than 2 meters away from the transmitter, the phase will change multiple 2π for the carrier frequency 1 GHz. It is reasonable to say that the phase probability density function is uniformly distributed in 2π range [1]

$$p_\varphi=\frac{1}{2\pi}. \qquad -\pi\le\varphi\le\pi. \qquad (4)$$

It is clear that the joint probability density function is the product of the probability density function of the phase and the probability density function of the amplitude. The phase and the amplitude are therefore independent.

Let us assume that the random complex field has the form

$$\Phi(\rho,\phi,z,f)=\Phi_r+j\Phi_i, \qquad (5)$$

where Φ_r and Φ_i are real and imaginary parts of the complex field, respectively. The joint probability density function of the complex field Φ is written as

$$p_2(\Phi_r,\Phi_i)=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(\Phi_r-m_r)^2+(\Phi_i-m_i)^2}{2\sigma^2}\right], \qquad (6)$$

where m_r and m_i are expected values of the real and imaginary parts of the complex field, respectively, and σ is the standard deviation. In polar coordinates

$$\left.\begin{aligned}\Phi_r&=R\cos\phi\\ \Phi_i&=R\sin\phi\end{aligned}\right\}. \qquad (7)$$

The Jacobian is $|J|=R$. Then the joint probability density function of the complex field Φ in polar coordinates is

$$p_2(R,\phi)=|J|\,p_2(\Phi_r,\Phi_i)=\frac{R}{2\pi\sigma^2}\exp\left[-\frac{R^2-2Rm\cos\psi+m^2}{2\sigma^2}\right], \qquad (8)$$

where

$$m_r\cos\phi+m_i\sin\phi=m\cos(\theta-\phi), \qquad (9)$$

$$\theta-\phi=\psi, \qquad (10)$$

$$m=\sqrt{m_r^2+m_i^2}, \qquad (11)$$

$$\tan\theta = \frac{m_i}{m_r}. \quad (12)$$

The probability density function for the random variable *R* is

$$p_1(R) = \int_{-\pi}^{\pi} p_2(R,\phi)d\phi = \frac{R}{\pi\sigma^2}\exp\left(-\frac{R^2+m^2}{2\sigma^2}\right)\int_0^{\pi}\exp\left(\frac{Rm\cos\psi}{\sigma^2}\right)d\psi. \quad (13)$$

By 9.6.16 of [2], we have

$$p_1(R) = \frac{R}{\sigma^2}\exp\left(-\frac{R^2+m^2}{2\sigma^2}\right)I_0\left(\frac{Rm}{\sigma^2}\right), \quad (14)$$

where $I_0(x)$ is the zeroth order modified Bessel function of the first kind. From (4.55) of [3], the probability density function for the random field amplitude *R* is Ricean. The parameter *m* denotes the peak amplitude of the dominant signal.

Casey has derived the expected value and standard deviation for the complex modal field in a straight rough-wall tunnel [4]. For normalized *Nlth* TE or TM mode the expected values are

$$m_r = J_N\left(\frac{p_{Nl}\rho}{a}\right)\cos(N\phi - k_{zNl}), \quad (15)$$

$$m_i = J_N\left(\frac{p_{Nl}\rho}{a}\right)\sin(N\phi - k_{zNl}), \quad (16)$$

where $J_N(x)$ is *Nth* order first kind of Bessel function, p_{Nl} is the *Nlth* mode normalized cutoff wave number, k_{zNl} is the Nlth mode wave number, and *a* is the average radius of the tunnel. It is clear that

$$m = J_N\left(\frac{p_{Nl}\rho}{a}\right). \quad (17)$$

From (17) the peak amplitude of signal *m* depends on radial variable ρ only, not the angular variable. The Nlth mode standard deviation for the TM case

$$\begin{aligned}\sigma_{Nl} = \Bigg\{ & \frac{|A_{Nl}|^2}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{Nk_z'}{a}R(k_z',\rho)J_N(p_{Nl})\right|^2 S_m(k_z')dk_z' \\ & + \frac{|A_{Nl}B_{Nl}|^2}{\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{R(k_z',\rho)J_N(p_{Nl})}{\hat{y}}\right|^2\frac{2\sigma Nk_z'p_{Nl}}{a^2}\left(k_z'k_{zNl} - \frac{p_{Nl}^2}{a^2}\right)S_m(k_z')dk_z' \quad (18) \\ & + \frac{|B_{Nl}|^2}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{R(k_z',\rho)J_N(p_{Nl})p_{Nl}}{\hat{y}a}\left(k_z'k_{zNl} - \frac{p_{Nl}^2}{a^2}\right)\right|^2 S_m(k_z')dk_z'\Bigg\}^{1/2},\end{aligned}$$

where

$$R(k_z',\rho) = -\frac{\hat{y}J_{m+N}(\lambda_{Nl}\rho)}{\lambda^2[J_{m+N}(\lambda a) + \lambda^2\sigma^2 J''_{m+N}(\lambda a)/2]}, \quad (19)$$

A_{Nl} and B_{Nl} are the *Nlth* constant coefficients for TM and TE modes, respectively, λ_0 and λ_{Nl} are the free space wavelength and *Nlth* mode wavelength, respectively. $\hat{y}$ is the free space admittance, and $S_m(k_z^{'})$ are power spectral densities such that

$$E[\Delta(\phi, z)\Delta(\phi^{'}, z^{'})] = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{jm(\phi-\phi')}\int_{-\infty}^{\infty} e^{-jk_z(z-z')}S_m(k_z)dk_z. \quad (20)$$

Here $\Delta(\phi, z)$ is the roughness of the wall. It is obvious that the standard deviation depends on the radial position as well. The *Nlth* mode standard deviation for the TE case is a rather complicated function of $m, N, \rho, k_z^{'}, k_{zNl}$. We shall not take the space to write it out.

CONCLUSION

The field propagating in caves or tunnels is a complex-valued Gaussian random process, by the central limit theorem. We assume that the phase probability density function is uniformly distributed over a range of extent 2π. Under this assumption the phase and amplitude of the field joint probability density function are independent and uncorrelated. We have shown that the probability density function for random field amplitude propagating in a straight rough wall tunnel or cave is Ricean. This tells us that there is a dominant signal component, such as a line-of-sight propagation path. In such a situation, random components arriving at different angles are superimposed on a stationary signal. At the output of an envelope detector, this has the effect of adding a dc component to the random multipath signal [5]. Since both expected value and standard deviation depend only on radial position, the probability density function for random field amplitude propagating in a straight rough wall tunnel or cave is a radially dependent function.

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